

On Murty-Simon Conjecture II

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Abstract

A graph is diameter two edge-critical if its diameter is two and the deletion of any edge increases the diameter. Murty and Simon conjectured that the number of edges in a diameter two edge-critical graph on n vertices is at most $\lfloor \frac{n^2}{4} \rfloor$ and the extremal graph is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. In the series papers [7–9], the Murty-Simon Conjecture stated by Haynes et al. is not the original conjecture, indeed, it is only for the diameter two edge-critical graphs of even order. In this paper, we completely prove the Murty-Simon Conjecture for the graphs whose complements have vertex connectivity ℓ , where $\ell = 1, 2, 3$; and for the graphs whose complements have an independent vertex cut of cardinality at least three.

Keywords: Murty-Simon Conjecture; diameter two edge critical graph; total domination edge critical graph

1 Introduction

All graphs considered in this paper are simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* of a vertex v in a graph G , denoted by $N_G(v)$, is the set of all the vertices adjacent to the vertex v , i.e., $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* of a vertex v in G , denoted by $N_G[v]$, is defined by $N_G[v] = N_G(v) \cup \{v\}$. For a subset $S \subseteq V$, the *neighborhood of the set S* in G is the set of all vertices adjacent to vertices in S , this set is denoted by $N_G(S)$, and the *closed neighborhood of S* by $N_G[S] = N_G(S) \cup S$. Let S and T be two subsets (not necessarily disjoint) of $V(G)$, $[S, T]$ denotes the set of edges of G with one end in S and the other in T , and $e_G(S, T) = |[S, T]|$. If every vertex in S is adjacent to each vertex in T , then we say that $[S, T]$ is full. If $S \subseteq V(G)$, and u, v are two nonadjacent vertices in G , then we say that xy is a *missing edge* in S (rather than “ uv is a missing edge in $G[S]$ ”).

The *complement* G^c of a simple graph $G = (V, E)$ is the simple graph with vertex set V , two vertices are adjacent in G^c if and only if they are not adjacent in G .

Given a graph G and two vertices u and v in it, the *distance* between u and v in G , denoted by $d_G(u, v)$, is the length of a shortest u - v path in G ; if there is no path connecting u and v , we define $d_G(u, v) = \infty$. The *diameter* of a graph G , denoted by $\text{diam}(G)$, is the maximum distance between any two vertices of G . Clearly, $\text{diam}(G) = \infty$ if and only if G is disconnected.

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A subset $S \subseteq V$ is called a *dominating set* (**DS**) of a graph G if every vertex $v \in V$ is an element of S or is adjacent to a vertex in S , that is, $N_G[S] = V$. The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G .

A subset $S \subseteq V$ is a *total dominating set*, abbreviated **TDS**, of G if every vertex in V is adjacent to a vertex in S , that is $N_G(S) = V$. Every graph without isolated vertices has a TDS, since V is a trivial TDS. The *total domination number* of a graph G , denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS in G . For the graph with isolated vertices, we define its total domination number to be ∞ . Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2].

Observation 1. Let G be a graph, for any vertex v and a TDS S in G . Then $S \cap N_G[v] \neq \emptyset$.

For two vertex subsets X and Y , we say that X *dominates* Y (*totally dominates* Y , respectively) if $Y \subseteq N_G[X]$ ($Y \subseteq N_G(X)$, respectively); sometimes, we also say that Y *is dominated by* X (*totally dominated by* X , respectively).

For three vertices $u, v, w \in V(G)$, the symbol $uv \rightarrow w$ means that $\{u, v\}$ dominates $G - w$, but $uw \notin E(G)$, $vw \notin E(G)$ and $uw \in E(G)$.

A graph G is said to be *diameter- d edge-critical* if $\text{diam}(G) = d$ and $\text{diam}(G - e) > \text{diam}(G)$ for any edge $e \in E(G)$. Glivak [5] proved the impossibility of characterization of diameter- d edge-critical graphs by finite extension or by forbidden subgraphs. Plesník [11] observed that all known minimal graphs of diameter two on n vertices have no more than $\lfloor \frac{n^2}{4} \rfloor$ edges. Independently, Murty and Simon (see [1]) conjectured the following:

Murty-Simon Conjecture. If G is a diameter-2 edge-critical graph on n vertices, then $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor$. Moreover, equality holds if and only if G is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Let G be a diameter-2 edge-critical graph on n vertices. Plesník [11] proved that $|E(G)| < 3n(n-1)/8$. Caccetta and Häggkvist [1] obtained that $|E(G)| < 0.27n^2$. Fan [3] proved the first part of the Murty-Simon Conjecture for $n \leq 24$ and for $n = 26$; and

$$|E(G)| < \frac{1}{4}n^2 + (n^2 - 16.2n + 56)/320 < 0.2532n^2$$

for $n \geq 25$. Füredi [4] proved the Murty-Simon Conjecture for $n > n_0$, where n_0 is not larger than a tower of 2's of height about 10^{14} .

A graph is *total domination edge critical* if the addition of any edge decrease the total domination number. If G is total domination edge critical with $\gamma_t(G) = k$, then we say that G is a *k - γ_t -edge critical graph*. Haynes et al. [10] proved that the addition of an edge to a graph without isolated vertices can decrease the total domination number by at most two. A graph G with the property that $\gamma_t(G) = k$ and $\gamma_t(G + e) = k - 2$ for every missing edge e in G is called a *k -supercritical graph*.

Theorem 1.1 (Hanson and Wang [6]). A nontrivial graph G is dominated by two adjacent vertices if and only if the diameter of G^c is greater than two.

Corollary 1. A graph G is diameter-2 edge-critical on n vertices if and only if the total domination number of G^c is greater than two but the addition of any edge in G^c decrease the total domination number to be two, that is, G^c is $K_1 \cup K_{n-1}$ or 3- γ_t -edge critical or 4-supercritical.

The complement of G is $K_1 \cup K_{n-1}$ if and only if G is $K_{1,n-1}$. Clearly, the Murty-Simon Conjecture holds for $K_{1,n-1}$.

The 4-supercritical graphs are characterized in [12].

Theorem 1.2. A graph H is 4-supercritical if and only if H is the disjoint union of two nontrivial complete graphs.

The complement of a 4-supercritical graph is a complete bipartite graph. The Murty-Simon Conjecture holds for the graphs whose complements are 4-supercritical, i.e., complete bipartite graphs.

Therefore, we only have to consider the graphs whose complements are 3- γ_t -edge critical.

For 3- γ_t -edge critical graphs, the bound on the diameter is established in [10].

Theorem 1.3. If G is a 3- γ_t -edge critical graph, then $2 \leq \text{diam}(G) \leq 3$.

Hanson and Wang [6] partition the family of 3- γ_t -edge critical graphs into two classes in terms of the diameter:

$$\mathfrak{G}_3 = \{G \mid G \text{ is a 3-}\gamma_t\text{-edge critical graph on } n \text{ vertices and } \text{diam}(G) = 3\}$$

$$\mathfrak{G}_2 = \{G \mid G \text{ is a 3-}\gamma_t\text{-edge critical graph on } n \text{ vertices and } \text{diam}(G) = 2\}$$

and proved the first part of the Murty-Simon Conjecture for the graphs whose complement are in \mathfrak{G}_3 . Recently, Haynes, Henning, van der Merwe and Yeo [7] proved the second part for the graphs whose complements are 3- γ_t -edge critical graphs with diameter three but only with even vertices. Also, Haynes et al. [9] proved the Murty-Simon Conjecture for the graphs of even order whose complements have vertex connectivity ℓ , where $\ell = 1, 2, 3$. Haynes, Henning and Yeo [8] proved the Murty-Simon Conjecture for the graphs whose complements are claw-free.

Remark 1. In the series papers [7–9], the Murty-Simon Conjecture stated by Haynes et al. is not the original conjecture, indeed, it is only for the diameter two edge critical graphs of even order. In this paper, we completely prove the Murty-Simon Conjecture for the graphs whose complements have vertex connectivity ℓ , where $\ell = 1, 2, 3$.

Let G be a 3- γ_t -edge critical graph. Then the addition of any edge e decrease the total domination number to be two, that is, $G + e$ is dominated by two adjacent vertices x and y ; we call such edge xy *quasi-edge* of e . Note that xy must contain at least one end of e . Clearly, quasi-edge of e may not be unique. If $xy \mapsto w$, then xy is quasi-edge of the missing edge xw , and also quasi-edge of missing edge yw ; conversely, if xy is quasi-edge of a missing edge, then there exists a unique vertex w such that $xy \mapsto w$.

From the definition of 3- γ_t -edge critical graph, we have the following frequently used observation.

Observation 2. If G is a 3- γ_t -edge critical graph and uv is a missing edge in it, then either

- (i) $\{u, v\}$ dominates G ; or
- (ii) there exists a vertex z such that $uz \mapsto v$ or $zv \mapsto u$.

2 Main results

The following fundamental result was observed by Hanson and Wang [6], also formally written by Haynes et al. [7].

Lemma 1. Let G be a graph on n vertices and (A, B) be a partition of its vertex set. If we can associate every missing edge in A or B with an edge in $[A, B]$ and this association is unique in sense that no two missing edges in A or B can associate with one edge in $[A, B]$, then $|E(G^c)| \leq |A| \times |B| \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. Moreover, if there exists an additional edge in $[A, B]$ which is not associate with any missing edge in A or B , then $|E(G^c)| < \left\lfloor \frac{n^2}{4} \right\rfloor$.

The following lemma is extracted from the proof in [7], but for the sake of completeness we present here a full self-contained proof.

Lemma 2. Let G be a $3\text{-}\gamma_t$ -edge critical graph on n vertices, and (A, B) be a partition of the vertex set $V(G)$. If, for every missing edge e in A and B , there exists a quasi-edge of e in $[A, B]$, then $|E(G^c)| \leq |A| \times |B| \leq \lfloor \frac{n^2}{4} \rfloor$. Moreover, if $|E(G^c)| = \lfloor \frac{n^2}{4} \rfloor$, then we have the following properties:

- (i) For every missing edge e in A and B , there exists precisely one quasi-edge of e in $[A, B]$; conversely, for every edge in $[A, B]$, it is the quasi-edge of a missing edge in A or B .
- (ii) If $u_1, u_2 \in A$ and $v_1, v_2 \in B$, $\{u_1v_1, u_2v_2\} \subseteq E(G^c)$ and $\{u_1v_2, u_2v_1\} \subseteq E(G)$, then $\{u_1u_2, v_1v_2\} \subseteq E(G)$.
- (iii) If u_1u_2 is a missing edge in A and $\deg_B(u_1) \geq \deg_B(u_2)$, then $N_B(u_1) = N_B(u_2) \cup \{y\}$, where y is the end (in B) of the quasi-edge of u_1u_2 . Similarly, if v_1v_2 is a missing edge in B and $\deg_A(v_1) \geq \deg_A(v_2)$, then $N_A(v_1) = N_A(v_2) \cup \{x\}$, where x is the end (in A) of the quasi-edge of v_1v_2 . Consequently, the missing edges in A (resp. in B) form a bipartite graph on A (resp. on B).

Proof. Suppose that uw is a missing edge in A , by the hypothesis, without loss of generality, there exists an edge uw in $[A, B]$ such that $uw \mapsto v$. Clearly, v is not dominated by $\{u, w\}$, and thus for any missing edge $e \neq uw$ in A or B , the edge uw is not a quasi-edge of e . Hence, for distinct missing edges e and e' in A , they have no common quasi-edges in $[A, B]$. Similarly, for distinct missing edges e and e' in B , they have no common quasi-edges in $[A, B]$.

It is easy to check that for any missing edge e in A and missing edge e' in B , they have no common quasi-edges in $[A, B]$. We associate every missing edge in A and B with its quasi-edge in $[A, B]$, by Lemma 1, it follows that $|E(G^c)| \leq |A| \times |B| \leq \lfloor \frac{n^2}{4} \rfloor$.

(ii) If $u_1u_2 \notin E(G)$, then both u_1v_2 and u_2v_1 are quasi-edge of u_1u_2 , a contradiction. Similarly, we can prove that $v_1v_2 \in E(G)$.

(iii) Let u_1u_2 be a missing edge in A . Suppose that $N_B(u_1) \not\subseteq N_B(u_2)$ and $N_B(u_2) \not\subseteq N_B(u_1)$. Choose a vertex $v_1 \in N_B(u_2) \setminus N_B(u_1)$ and a vertex $v_2 \in N_B(u_1) \setminus N_B(u_2)$, then $\{u_1v_1, u_2v_2\} \subseteq E(G^c)$ and $\{u_1v_2, u_2v_1\} \subseteq E(G)$, by (ii), we have $u_1u_2 \in E(G)$, a contradiction. Hence $N_B(u_1) \supseteq N_B(u_2)$. If $|N_B(u_1) \setminus N_B(u_2)| \geq 2$, then there are at least two quasi-edge of the missing edge u_1u_2 , a contradiction. Therefore, $N_B(u_1) = N_B(u_2) \cup \{y\}$. Similarly, we can prove that $N_A(v_1) = N_A(v_2) \cup \{x\}$, if v_1v_2 is a missing edge in B .

In the graph formed by the missing edges in A , one part X is the vertices of degree odd in B , and the other part Y is the vertices of degree even in B . For any missing edge uw , $\deg_B(u)$ and $\deg_B(w)$ differ by exactly one, so one is odd and the other is even, and hence uw has one end in X and the other in Y , then the graph is bipartite. Similarly, the graph formed by the missing edges in B is a bipartite graph. \square

To settle the Murty-Simon Conjecture, the remaining graphs to be verified are ones whose complements are in \mathfrak{G}_2 . We show that the conjecture holds if a condition in terms of independent cut is satisfied.

Theorem 2.1. Let G be a $3\text{-}\gamma_t$ -edge critical graph on n vertices with $\delta(G) \geq 3$. If G has an independent vertex cut of cardinality at least three, then $|E(G^c)| < \lfloor \frac{n^2}{4} \rfloor$.

Proof. First, we prove the following asseration:

Asseration. There exists an independent vertex cut S and a component K of $G - S$ such that every vertex in K dominates $V(K) \cup S$.

Proof. If G has a vertex v such that $N_G(v)$ is independent, then let $S = N_G(v)$ and $K = \{v\}$, we are done. So we may assume that there is no such vertex. Let S be an independent vertex cut of cardinality at least three and G_1 be a component of $G - S$, and G_2 be the union of the other components. Moreover, by the above argument, we may assume that $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$.

Assume that there exists a vertex $v \in S$, a vertex $w_1 \in V(G_1)$ and a vertex $w_2 \in V(G_2)$ such that $\{vw_1, vw_2\} \subseteq E(G^c)$. Since $\{w_1, w_2\}$ does not dominate v , by Observation 1, there exists a vertex w such that either w_1w is an edge and $w_1w \mapsto w_2$ or ww_2 is an edge and $ww_2 \mapsto w_1$. In the former case, $w \in V(G_1)$,

but $\{w_1, w\}$ does not dominate $V(G_2) \setminus \{w_2\}$ for $|V(G_2)| \geq 2$, a contradiction; a similar contradiction can be obtained for the latter case. Therefore, for every vertex v in S , it dominates G_1 or G_2 .

Suppose that G_1 is not dominated by $v_1 \in S$ and G_2 is not dominated by $v_2 \in S$. By the previous argument, $v_1 \neq v_2$ and thus v_1 dominates G_2 and v_2 dominates G_1 . Since $\{v_1, v_2\}$ does not dominate $S \setminus \{v_1, v_2\}$ (note that $S \setminus \{v_1, v_2\} \neq \emptyset$ for S is independent and $|S| \geq 3$), by Observation 1, there exists a vertex v' such that v_1v' is an edge and $v_1v' \mapsto v_2$ or $v'v_2$ is an edge and $v'v_2 \mapsto v_1$. Without loss of generality, assume that the former case holds. Since v_1 does not dominate G_1 , the vertex v' must be in $V(G_1)$ and $v'v_2 \notin E(G)$, which contradicts the fact that v_2 dominates G_1 . Therefore, without loss of generality, we may assume that G_1 is dominated by every vertex in S .

Next, we show that G_1 is complete. Otherwise, there is a missing edge u_1u_2 in $V(G_1)$. Since $\{u_1, u_2\}$ does not dominate G_2 , there exists a vertex u such that $u_1u \mapsto u_2$ or $uu_2 \mapsto u_1$. Without loss of generality, assume that $u_1u \mapsto u_2$. We have known $u_1u \in E(G)$ and $uu_2 \notin E(G)$, so $u \in V(G_1)$. But $\{u_1, u\}$ does not dominate $V(G_2)$, which is a contradiction. Therefore, G_1 is complete. Let $K = G_1$, we complete the proof. \square

Let $A = V(K) \cup S$ and $B = V \setminus A$. For any missing edge xy in A , indeed xy is a missing edge in S by the assertion. Both x and y dominates K , the quasi-edge of xy must have one end in B , i.e., its quasi-edges lies in $[A, B]$. For any missing edge $x'y'$ in B , the closed neighborhood of every vertex in K is contained in A , then quasi-edges of $x'y'$ must have one end in A . By Lemma 2,

$$|E(G^c)| \leq |A| \times |B| \leq \left\lfloor \frac{n^2}{4} \right\rfloor. \quad (2.1)$$

If $|E(G^c)| = \left\lfloor \frac{n^2}{4} \right\rfloor$, then the missing edges in A form a bipartite graph by Lemma 2 (iii), but indeed it is a clique with at least three vertices, a contradiction. \square

Theorem 2.2. If G is a $3\text{-}\gamma_t$ -edge critical graph on n vertices and with connectivity one, then $|E(G^c)| < \left\lfloor \frac{n^2}{4} \right\rfloor$.

Proof. If $\text{diam}(G) = 3$, then we are done in [13]. So we may assume that $\text{diam}(G) = 2$. If v is a cut vertex of G , then v dominates G , and hence v and one of its neighbor totally dominates G , this contradicts the fact that $\gamma_t(G) = 3$. \square

Remark 2. For the connectivity $\ell = 2$, Haynes et al. give a proof in [9], indeed, their proof covers the graphs of odd order, but they used the result about claw-free case in the proof, so we give a direct proof here.

Theorem 2.3. If G is a $3\text{-}\gamma_t$ -edge critical graph on n vertices and with connectivity two, then $|E(G^c)| < \left\lfloor \frac{n^2}{4} \right\rfloor$.

Proof. If $\text{diam}(G) = 3$, then we are done in [13]. So we may assume that $\text{diam}(G) = 2$. Let $\{x, y\}$ be a minimum vertex cut. A vertex in $G - \{x, y\}$ is called *strong* if it joins to both x and y , and *weak* if it joins to one of x and y .

We state the following properties, they are very simple, so we omit their proofs, the readers can also find the proofs in [9].

- (i) $\{x, y\}$ dominates G and every vertex in $G - \{x, y\}$ is either strong or weak;
- (ii) x and y are nonadjacent;
- (iii) the strong vertices in the same component of $G - \{x, y\}$ forms a clique;
- (iv) there is at most one component of $G - \{x, y\}$ containing weak vertices;

- (v) the set of weak vertices is a clique;
- (vi) there are precisely two components of $G - \{x, y\}$.

Let G_1 and G_2 be the two components of $G - \{x, y\}$. Without loss of generality, assume that all the vertices in G_1 are strong. Let $A = V(G_1) \cup \{x, y\}$ and $B = V(G_2)$.

The set $\{x, y\}$ is a minimum vertex cut, there are at least two edges in $[A, B]$. If G_2 is complete, then there are only one missing edge (say xy) in A and B , and thus $|E(G^c)| \leq |A| \times |B| - 1 < \lfloor \frac{n^2}{4} \rfloor$, we are done. So we may assume that G_2 is not complete. Let uv be a missing edge in G_2 . By the previous assertions, assume that u is a strong vertex and v is a weak vertex. Since $\{u, v\}$ does not dominate G_1 , there exists a vertex w such that $uw \mapsto v$ or $wv \mapsto u$. In both cases, the vertex w has to dominate G_1 , it follows that $w \in \{x, y\}$. If $wv \mapsto u$, then $wu \notin E(G)$, a contradiction. Then $uw \mapsto v$ and uw is the quasi-edge of uv . Therefore, the quasi-edges of missing edges in G_2 are between $\{x, y\}$ and the strong vertices of G_2 . If there are at least two weak vertices in G_2 , then $|E(G^c)| \leq |A| \times |B| - 1 < \lfloor \frac{n^2}{4} \rfloor$, we are done. So there exists only one weak vertex, say v , in G_2 . Assume that $yv \in E(G)$. Therefore, for any missing edge uv in B , xu is the quasi-edge of uv . There are two edges yu, yv are not the quasi-edge of any missing edge in B , but there exist only one missing edge in A , so $|E(G^c)| \leq |A| \times |B| - 1 < \lfloor \frac{n^2}{4} \rfloor$. \square

Theorem 2.4. If G is a $3\text{-}\gamma_t$ -edge critical graph on n vertices and with connectivity 3, then $|E(G^c)| < \lfloor \frac{n^2}{4} \rfloor$.

Proof. If $\text{diam}(G) = 3$, then the theorem was proved in [13]. So we may assume that $\text{diam}(G) = 2$. Let $\{x, y, z\}$ be a minimum vertex cut of G .

Claim 1. There are precisely two components of $G - \{x, y, z\}$.

Proof. See [9]. \square

Let G_1 and G_2 be the two components of $G - \{x, y, z\}$, and let V_1 and V_2 be the vertex set of G_1 and G_2 , respectively.

Claim 2. For any two vertices v, v' in the same component of $G - \{x, y, z\}$, if both v and v' dominate $\{x, y, z\}$, then $vv' \in E(G)$.

Proof. Suppose that $vv' \notin E(G)$. Since $\{v, v'\}$ does not dominate the other component, there exists a vertex w such that $vw \mapsto v'$ or $wv' \mapsto v$. In order to dominate the other component, $w \in \{x, y, z\}$, but both v and v' dominates $\{x, y, z\}$, a contradiction. \square

It is easy to check the following asseration:

Claim 3. For any vertices $v_1 \in V_1$ and $v_2 \in V_2$, we have either $N_G(v_1) \cap \{x, y, z\} \neq N_G(v_2) \cap \{x, y, z\}$ or both v_1 and v_2 dominates $\{x, y, z\}$.

For $i = 1, 2$, let S_i^* be the set of vertices in $\{x, y, z\}$ which dominates V_i .

Claim 4. We may assume that $|S_1^* \cup S_2^*| = 3$.

Proof. We may assume, on the contrary, that there exists a vertex, say z , such that $\{zv_1, zv_2\} \subseteq E(G^c)$, where $v_1 \in V_1$ and $v_2 \in V_2$. Since $\deg_G(v_i) \geq 3$ and $zv_i \notin E(G)$, we have $|V_i| \geq 2$ for $i = 1, 2$. Since $d_G(v_1, v_2) = 2$, v_1 and v_2 have a common neighbor, say x , in $\{x, y, z\}$. Since $\{v_1, v_2\}$ does not dominate z , there exists a vertex w such that $v_1w \mapsto v_2$ or $wv_2 \mapsto v_1$. Without loss of generality, assume that $v_1w \mapsto v_2$. Since $v_1w \in E(G)$ and $wv_2 \notin E(G)$, it follows that $w = y$ and $yz \in E(G)$ and y dominates V_2 except v_2 . Since $\text{diam}(G) = 2$ and $\{yv_2, zv_2\} \subseteq E(G^c)$, it yields that x dominates V_1 . Hence $xy \notin E(G)$, for otherwise $\{x, y\}$ totally dominates G , which is a contradiction. If uv is a missing edge in V_1 , then there exists a vertex w' such that $uw' \mapsto v$ or $w'v \mapsto u$, in both cases, w' dominates V_2 , so $w' = x$, but $\{xu, xv\} \subseteq E(G)$,

a contradiction. Therefore, V_1 is a clique. The vertex v_2 has only one neighbor x in $\{x, y, z\}$, by Claim 3, for any vertex in V_1 , it has one neighbor in $\{y, z\}$, and thus $\{y, z\}$ dominates V_1 .

Suppose that v_2v' is a missing edge in V_2 . Consider $G + v_1v'$. Since $\{v_1, v'\}$ does not dominate v_2 , there exists a vertex w^* such that $v_1w^* \mapsto v'$ or $w^*v' \mapsto v_1$. If $w^*v' \mapsto v_1$, then $w^*v_1 \notin E(G)$ and $w^*v' \in E(G)$, it follows that $w^* = z$, but $\{z, v'\}$ does not dominate v_2 , a contradiction. So we may assume that $v_1w^* \mapsto v'$, then $v_1w^* \in E(G)$ and $w^*v_2 \in E(G)$, so $w^* = x$ and x dominates V_2 except v' . Consider $G + v_2v'$, there exists a vertex w in $\{x, y, z\}$ such that $v_2w \mapsto v'$ or $wv' \mapsto v_2$. If $v_2w \mapsto v'$, then $wv_2 \in E(G)$ and $w = x$, but $\{x, v_2\}$ does not dominate y , a contradiction. If $wv' \mapsto v_2$, then $wv' \in E(G)$ and $wv_2 \notin E(G)$, so $w = y$, but $\{y, v'\}$ does not dominate x , a contradiction. Hence we may assume that v_2 dominates V_2 .

For any missing edge $u'v'$ in V_2 , indeed, it is a missing edge in $N_{G_2}(y)$. Consider $G + u'v'$, quasi-edges of $u'v'$ lies in $\{\{x\}, V_2 \setminus \{v_2\}\}$.

Let $A = V_1 \cup \{x, y\}$ and $B = V_2 \cup \{z\}$. We associate every missing edge $u'v'$ in V_2 with one of its quasi-edge in $\{\{x\}, V_2 \setminus \{v_2\}\}$; associate the missing edge in $\{\{z\}, V_2\}$ with edges in $\{\{y\}, V_2\}$; associate the missing edge xy with yz . In addition, there is an additional edge xv_2 , therefore, $|E(G^c)| < \lfloor \frac{n^2}{4} \rfloor$ by Lemma 1. \square

Claim 5. We may assume that $|V_1| = 1$ and V_2 is not a clique.

Proof. By Claim 4, we have $S_1^* \cup S_2^* = \{x, y, z\}$, without loss of generality, assume that $\{x, y\} \subseteq S_1^*$. We may assume, on the contrary, that $|V_1| \geq 2$. Let v_1 be a neighbor of z in V_1 , hence v_1 dominates $\{x, y, z\}$. Let Q be the set of vertices in V_2 which does not dominate V_2 . For any vertex $v \in Q$, since $\{v_1, v\}$ does not dominate V_2 , there exists a vertex w_v such that $v_1w_v \mapsto v$ or $w_vv \mapsto v_1$. If $w_vv \mapsto v_1$, then $w_v \notin \{x, y, z\}$ and $w_v \in V_2$, but $\{w_v, v\}$ does not dominate $V_1 \setminus \{v_1\}$, which is a contradiction. Therefore, for any vertex $v \in Q$, there exists a vertex w_v in $\{x, y, z\}$ such that $v_1w_v \mapsto v$, and thus w_v dominates V_2 except v . Note that for distinct vertices v and v' in Q , $w_v \neq w_{v'}$, therefore, $|Q| \leq 3$.

If $|Q| = 3$, then $|V_2| \geq 4$, for otherwise V_2 is disconnected, a contradiction. Hence, for every vertex $x' \in V_2 \setminus Q$, it dominates $V_2 \cup \{x, y, z\}$, consequently, $\{x, x'\}$ totally dominates G , a contradiction.

If $|Q| = 2$, then $|V_2| \geq 3$, for otherwise V_2 is disconnected, a contradiction. Suppose that z dominates V_1 . By Claim 2, V_1 is a clique. Without loss of generality, assume that for vertices $v, v' \in Q$, we have $w_v = x$ and $w_{v'} = y$. Let $A = V_1 \cup \{x, y\}$ and $B = V_2 \cup \{z\}$. We associate the missing edge vv' in V_2 with zv_1 , associate missing edges between z and V_2 with edges in $\{\{y\}, V_2\}$. Now, there are at least two edges in $\{\{x\}, V_2\}$ but there are at most one missing edge (say, xy) in $V_1 \cup \{x, y\}$, hence $|E(G^c)| < \lfloor \frac{n^2}{4} \rfloor$ by Lemma 1, we are done. So we may assume that z dominates V_2 , then we have $\{w_v, w_{v'}\} = \{x, y\}$. Also let $A = V_1 \cup \{x, y\}$ and $B = V_2 \cup \{z\}$. For any missing edge e in V_1 , quasi-edges of e lies in $\{\{z\}, V_1\}$, we associate the missing edge e with one of its quasi-edges, associate the missing edge in V_2 with one edge in $\{\{x, y\}, V_2\}$, associate one edge in $\{\{x, y\}, V_2\}$ with the possible missing edge xy . In the final, there are at least $2|V_2| - 2 - 2 \geq 2$ additional edges, hence $|E(G^c)| < \lfloor \frac{n^2}{4} \rfloor$ by Lemma 1.

Next, we consider the case $|Q| = 0$, i.e., V_2 is a clique. If z dominates V_2 , then $V_2 \cup \{z\}$ is a clique. For any missing edge in V_1 , its quasi-edges lies in $\{\{z\}, V_1\}$. There are at least two edges in $\{\{x, y\}, V_2\}$ and at most one other missing edge (say xy) in $V_1 \cup \{x, y\}$, then there are at least one additional edge, hence $|E(G^c)| \leq |V_1 \cup \{x, y\}| \times |V_2 \cup \{z\}| - 1 < \lfloor \frac{n^2}{4} \rfloor$. If z does not dominate V_2 , then it dominates V_1 and V_1 is a clique by Claim 2.

Now, if $|V_1| \geq 2$, then we may assume that V_1 is dominated by every vertex in $\{x, y, z\}$, and both V_1 and V_2 are clique. Clearly, if $|V_1| = 1$, then V_1 is also dominated by every vertex in $\{x, y, z\}$. In order to prove the claim by contradiction, we may assume that both V_1 and V_2 are cliques regardless the size of V_1 . Let $A' = V_1 \cup \{x, y, z\}$ and $B' = V_2$. There are at least three edges between A' and B' since $\{x, y, z\}$ is a minimum vertex cut, and there are at most three missing edges in A' . If $|E(G^c)| = \lfloor \frac{n^2}{4} \rfloor$, then $\{x, y, z\}$ is independent and $|\{\{x, y, z\}, V_2\}| = 3$, the subgraph formed by the missing edges in A' contains a triangle, which contradicts with Lemma 2. \square

For every missing edge in V_2 , we associate it with an unique quasi-edge of it, and denote this set by Q_e .

Claim 6. We may assume that there are at least two edges in $\{x, y, z\}$.

Proof. Suppose that the subgraph induced by $\{x, y, z\}$ has at most one edge, without loss of generality, let z be an isolated vertex in this subgraph. Let Q be the set of vertices in V_2 which dominates $\{x, y, z\}$. Then Q is a clique by Claim 2. Let $R = V_2 \setminus Q$, $B = \{z\} \cup R$ and $A = V \setminus B$.

Let v be an arbitrary vertex in R . If $zv \in E(G)$, then $zv \notin Q_e$, for otherwise, v has to dominate $\{x, y\}$ and $v \in Q$, a contradiction. If $zv \notin E(G)$, then for every edge in $[\{x, y\}, \{v\}]$ (note that $[\{x, y\}, \{v\}] \neq \emptyset$), it can not belong to Q_e , for otherwise, v has to dominate z and $zv \in E(G)$, a contradiction. Hence, for any vertex v in R , there is at least one edge in $[\{v\}, \{x, y, z\}]$ such that it is not in Q_e .

For any missing edge e in R , quasi-edges of e lies in $[\{x, y\}, R]$. If $xy \notin E(G)$, then R is a clique, moreover, $[Q, R] \neq \emptyset$ since V_2 is not a clique and V_2 is connected. So, if $xy \notin E(G)$, then we associate xy with one edge in $[Q, R]$. If zv is a missing edge in B , we associate an edge in $[\{x, y\}, \{v\}] \setminus Q_e$ with zv . We associate the missing edges in $[\{v_1\}, Q]$ with edges in $[\{z\}, Q]$. There is an additional edge zv_1 , and hence $|E(G^c)| < \lfloor \frac{n^2}{4} \rfloor$ by Lemma 1. \square

Let $B = V_2$ and $A = V \setminus B$. Without loss of generality, let xyz be a path, xz may be a missing edge.

Since V_2 is not a clique, $|V_2| \geq 2$; moreover, $|V_2| \geq 3$ since V_2 is connected. If $|V_2| = 3$, then V_2 has only one missing edge. But $|[A, B]| \geq 3$, and there are at most two missing edges in A and B , so $|E(G^c)| < \lfloor \frac{n^2}{4} \rfloor$, a contradiction. Obviously, the degree of y is at least four. If $|V_2| = 4$, then $|E(G^c)| = \binom{8}{2} - |E(G)| \leq 28 - 13 < \lfloor \frac{n^2}{4} \rfloor$. So we may assume that $|V_2| \geq 5$.

Case 1. $xz \notin E(G)$.

We may assume that there is at most one edge in $[\{x, y, z\}, V_2]$ which is not in Q_e , for otherwise we associate one of them with the missing edge xz , there is at least one additional edge, hence $|E(G^c)| < \lfloor \frac{n^2}{4} \rfloor$ by Lemma 1.

The set $\{x, y, z\}$ is a minimum vertex cut, so each of x and z has at least one neighbor in V_2 . Without loss of generality, assume that $xw \in [\{x\}, V_2]$ such that it is in Q_e and $xw \mapsto w_1$. Hence $wz \in E(G)$ since w has to dominate z . Suppose further that wz is also in Q_e and $wz \mapsto w_2$. Note that $w_1 \neq w_2$ and $\{zw_1, xw_2\} \subseteq E(G)$. But xw_2 is not in Q_e since $\{x, w_2\}$ does not dominate z , and zw_1 is also not in Q_e since $\{z, w_1\}$ does not dominate x . Now, there are at least two edges in $[\{x, y, z\}, V_2]$ which are not in Q_e , a contradiction. Hence, $zw \in E(G)$ but $zw \notin Q_e$ and every edge in $[\{x, y, z\}, V_2] \setminus \{zw\}$ is in Q_e . By the previous argument, we have $[\{x, z\}, V_2] = \{xw, zw\}$. As $\text{diam}(G) = 2$, every vertex in V_2 has at least one neighbor in $\{x, y, z\}$, so $[\{y\}, V_2 \setminus \{w\}]$ is full and $[\{y\}, V_2 \setminus \{w\}] \subseteq Q_e$. For every edge yw' in $[\{y\}, V_2 \setminus \{w\}]$, it is a quasi-edge of missing edge in V_2 , hence $yw' \mapsto w$ and $ww' \notin E(G)$, moreover, w is isolated in V_2 , which is a contradiction.

Case 2. $xz \in E(G)$ and hence A is a clique.

We associate every missing edge in B with a unique quasi-edge in Q_e , hence, $4(n-4) \geq |E(G^c)|$. If $|E(G^c)| \geq \lfloor \frac{n^2}{4} \rfloor$, then $n = 9$ and $Q_e = [A, B]$ and Lemma 2 (i)–(iii) holds.

For any vertex in V_2 , it does not dominate V_2 , otherwise, choose a neighbor of it in $\{x, y, z\}$, we obtain a two vertex set totally dominates G , a contradiction.

Let

$$X = \{v \in V_2 \mid v \text{ has an odd number of neighbors in } \{x, y, z\}\}$$

and

$$Y = \{v \in V_2 \mid v \text{ has an even number of neighbors in } \{x, y, z\}\}.$$

Then $X \cup Y = V_2$, and the missing edges in V_2 form a bipartite graph H with bipartition (X, Y) by Lemma 2. Let $m^* = \min\{|X|, |Y|\}$. Since $|Q_e| = |[A, B]| \geq |V_2| = 5$, there are at least five missing edges in V_2 , i.e.,

H has at least five edges, so $m^* = 2$. Hence, there are at least $|X| + 2|Y| \geq |V_2| + m^* = 7$ edges in $[A, B]$, but there are at most $|X| \times |Y| = 6$ edges in H , i.e., there are at most 6 missing edges in V_2 , a contradiction. \square

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